Stat 155 Lecture 19 Notes

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1 Models for Transferable Utility

1.1 Nash's bargaining theorem and relationship to transferable utility

Last time, we mentioned Nash's bargaining theorem.

Theorem 1.1. *There is a unique function F satisfying Nash's bargaining axioms. It is the function that takes S and d and returns the unique solution to the optimization problem*

$$\max_{x_1,x_2}(x_1 - d_1)(x_2 - d_2)$$

subject to the constraints

$$\begin{aligned} &x_1 \geq d_1 \\ &x_2 \geq d_2 \\ &(x_1, x_2) \in S. \end{aligned}$$

We are talking about games with nontransferable utility, but this is also related to games with transferable utility.

Example 1.1. Consider a transferable utility game with disagreement point d and cooperative strategy with total payoff σ. Then the convex set S is the set of convex combinations of lines $\{(a_{i,j} + p, b_{i,j} - p) : p \in \mathbb{R}\}$. To maximize $(x_1 - d_1)(x_2 - d_2)$, we set $x_2 = \sigma - x_1$ and choose x_1 to maximize

$$(x_1 - d_1)(\sigma - x_1 - d_2) = -x_1^2 + (\sigma - d_2 + d_1)x_1 - d_1(\sigma - d_2).$$

This gives $x_1 = (\sigma - d_2 + d_1)/2$.

The Nash solution is unique. See the text for a slick proof. The Nash solution satisfies the bargaining axioms:

1. Pareto optimality: increasing, say, x_1 increases $(x_1 - d_1)(x_2 - d_2)$.

2. Symmetry: You can check that this follows from uniqueness of the solution.

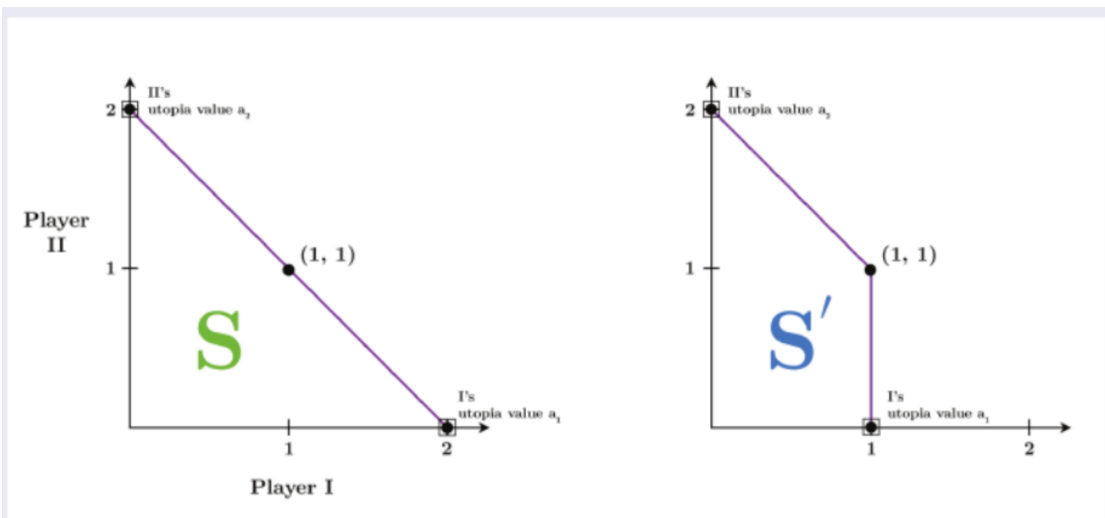
3. Affine covariance: $\alpha_1 x_1 + \beta_1 - (\alpha_1 d_1 + \beta_1) = \alpha_1(x - d_1)$.

4. Independence of irrelevant attributes: A maximizer in S that belongs to R is still a maximizer in $R \subseteq S$.

Here is the idea of the proof of the theorem.

Proof. Any bargaining solution that satisfies the axioms is the Nash solution. For S and d, if the Nash solution is a, find the affine function so that $\psi(a) = (1,1)$ and $\psi(d) = (0,0)$. If the Nash solution is $a = (1,1)$ and $d = (0,0)$, then the convex hull of S and its reflection are in $\{x_1 + x_2 \leq 2\}$, so any symmetric, optimal F returns $(1,1)$ for this convex hull,and hence, by IIA, for S. □

The affine covariance property is not always easily evident. Consider the following region S, and a region S' that is the image of S under and affine transformation.



Here, it seems like Player 2 should have an advantage somehow, but the Nash solution is $(1,1)$ for the region S'. Is this how players would choose a solution in real life?

1.2 Multiplayer transferable utility games

1.2.1 Allocation functions and Gillies' core

Example 1.2. A customer in a marketplace is willing to buy a pair of gloves for \$100. There are three players, one with right gloves and two with only left gloves, and they need to agree on who sells their glove and how to split the \$100. This is more complicated than a two-player game: the players can form coalitions. Who holds the power and what's fair depends on how the different subsets of players depend on other players and contribute to the payoff.

Definition 1.1. For each subset S of players, let $v(S)$ be the total value that would be available to be split by that subset of players no matter what the other players do. We call v a *characteristic function.*

Example 1.3. In our glove example, we have the following characteristic function:

$$v(\{1,2,3\}) = v(\{1,2\}) = v(\{1,3\}) = 100,$$

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2,3\}) = v(\varnothing) = 0.$$

Definition 1.2. An *allocation function* is a map from a characteristic function v for n players to a vector $\psi(v) \in \mathbb{R}^n$. This is the payoff that is allocated to the n players.

What properties should an allocation function have?

1. Efficiency: The total payoff gets allocated. That is,

$$\sum_{i=1}^{n} \psi_i(v) = v(\{1, \ldots, n\}).$$

2. Stability: Each coalition is allocated at least the payoff it can obtain on its own. For each $S \subseteq \{1, \ldots, n\}$,

$$\sum_{i \in S} \psi_i(v) \geq v(S).$$

The conditions are called *Gillies' core.*[1]

Example 1.4. Let's go back to the left and right gloves example.

$$\sum_{i=1}^{3} \psi_i(v) = v(\{1,2,3\}) = 100$$

$$\psi_1(v) + \psi_2(v) \geq 100, \qquad \psi_1(v) + \psi_3(v) \geq 100.$$

There is one solution: $\psi_1(v) = 100$.

[1]Donald B Gillies is a Canadian-born mathematician, game theorist, and computer scientist at the University of Illinois at Urbana-Champaign.

Example 1.5. Consider a game where any pair of gloves sells for \$1. The characteristic function is

$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1,$$
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\varnothing) = 0.$$

Then

$$\sum_{i=1}^{3} \psi_i(v) = v(\{1,2,3\}) = 1,$$
$$\psi_1(v) + \psi_2(v) \geq 1, \qquad \psi_1(v) + \psi_3(v) \geq 1, \qquad \psi_2(v) + \psi_3(v) \geq 1.$$

There are no solutions!

Example 1.6. Consider a game where single gloves sell for \$1, pairs sell for \$10, and triples sell for \$100. The characteristic function is

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 1,$$
$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 10,$$
$$v(\{1,2,3\}) = 100.$$

Then

$$\sum_{i=1}^{3} \psi_i(v) = v(\{1,2,3\}) = 100,$$
$$\psi_1(v) \geq 1, \qquad \psi_2(v) \geq 1, \qquad \psi_3(v) \geq 1$$
$$\psi_1(v) + \psi_2(v) \geq 10, \qquad \psi_1(v) + \psi_3(v) \geq 10, \qquad \psi_2(v) + \psi_3(v) \geq 10,$$
$$\psi_1(v) + \psi_2(v) + \psi_3(v) \geq 100.$$

There are many solutions!

As we can see, Gillies' core, while reasonable, may not be the most accurate model.

1.2.2 Shapley's axioms for allocation functions

Here are Shapley's axioms for allocation functions.

1. Efficiency: $\sum_{i=1}^{n} \psi_i(v) = v(\{1,\dots,n\})$.
2. Symmetry: If, for all $S \subseteq \{1,\dots,n\}$ and $i,j \notin S$, $v(S \cup \{i\}) = v(S \cup \{j\})$, then $\psi_i(v) = \psi_j(v)$.
3. No freeloaders: For all i, if for all $S \subseteq \{1,\dots,n\}$, $v(S \cup \{i\}) = v(S)$, then $\psi_i(v) = 0$.
4. Additivity: $\psi_i(v+u) = \psi_i(v) + \psi_i(u)$.

Theorem 1.2 (Shapley)**.** *Shapley's axioms uniquely determine the allocation ψ.*